

# REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY $G$ -BROWNIAN MOTION WITH NONLINEAR RESISTANCE

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**ABSTRACT.** In this paper, we study the uniqueness and existence of solutions of RGSDEs with nonlinear resistance under an integral-Lipschitz condition of coefficients. Moreover we obtain the comparison theorem for RGSDEs with nonlinear resistance.

## 1. INTRODUCTION

In classical framework, the diffusion processes with reflecting boundaries were introduced by Skorokhod [28, 29] in 1960s. After that, many works related to reflected solutions to SDEs and BSDEs have been done. El Karoui [3], El Karoui and Chaleyat-Maurel [4] and Yamada [32] studied scalar valued reflected SDEs on a half-line. For multidimensional case, Stroock and Varadhan [30] obtained the existence of weak solutions to reflected SDEs on a smooth domain which was extended to a convex domain by Tanaka [31] and a non convex domain by Lions and Sznitman [15]. On the other hand, the solvability of reflected BSDEs was first obtained by El Karoui et al.[5]. Then many corresponding results for reflected BSDEs have been established by Gegout-Petit and Pardoux [7], Ramasubramanian [26] and Hu and Tang [8], etc.. In particular, Qian and Xu [25] obtained the existence and uniqueness of solutions of reflected BSDEs with nonlinear resistance under a Lipschitz condition.

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng systemically established a time-consistent fully nonlinear expectation theory (see [19, 20, 22]). As a typical and important case, Peng introduced the  $G$ -expectation theory (see [23, 24] and the references therein) in 2006. In the  $G$ -expectation framework ( $G$ -framework for short), the notion of  $G$ -Brownian motion and the corresponding stochastic calculus of Itô's type were established. On that basis, Gao [6] and Peng [23] studied the existence and uniqueness of the solution of  $G$ -SDE under a standard Lipschitz condition. For a recent account and development of this theory we refer the reader to [9, 10, 11, 12, 14, 16, 17, 18, 21, 33]. Recently, Lin [13] obtained the existence and uniqueness of the solution of  $G$ -SDE with reflecting boundary. Compared to Lin [13], we consider an integral-Lipschitz condition of coefficients and also the increasing process  $K$  contributes to the coefficients, namely the following scalar valued RGSDE with nonlinear resistance:

$$(1.1) \quad \begin{cases} X_t = x + \int_0^t f_s(X_s, K_s)ds + \int_0^t h_s(X_s, K_s)d\langle B \rangle_s + \int_0^t g_s(X_s, K_s)dB_s + K_t, \text{ q.s., } 0 \leq t \leq T; \\ X_t \geq S_t; \int_0^T (X_t - S_t)dK_t = 0, \end{cases}$$

where  $\langle B \rangle$  is the quadratic variation process of  $G$ -Brownian motion  $B$ , and  $K$  is an increasing process which pushes the solution  $X$  upwards to be remaining above the obstacle  $S$  in a

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*Key words and phrases.*  $G$ -Brownian motion,  $G$ -expectation, reflected  $G$ -stochastic differential equations, nonlinear resistance, comparison theorem.

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The author's research was partially supported by China Scholarship Council, NSF (No. 10921101) and by the 111Project (No. B12023).

minimal way. The aim of this paper is to study the existence and uniqueness of solutions to the above RGSDEs with nonlinear resistance in the sense of “quasi-surely” defined by Denis et al. [2]. The main idea is to estimate in some sense simultaneously the solution  $X$  and the increasing process  $K$  from which the uniqueness result follows and a solution in  $M_G^p([0, T])$  to (1.1) can be constructed by fixed-point iteration. To establish the comparison theorem, we use the extended  $G$ -Itô formula in Lin [13].

This paper is organized as follows: Section 2 introduces some notations and results in  $G$ -framework which is necessary for what follows, while section 3 is our main results.

## 2. PRELIMINARIES

The main purpose of this section is to recall some preliminary results in  $G$ -framework which are needed in the sequel. More details can be found in Denis et al [2], Li and Peng [12], Lin [13, 14] and Peng [23].

Denote by  $\Omega = C_0^d(\mathbb{R}^+)$  the space of all  $\mathbb{R}^d$ -valued continuous paths  $(\omega_t)_{t \in \mathbb{R}^+}$ , with  $\omega_0 = 0$ , equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t \in [0, i]} |\omega_t^1 - \omega_t^2| \wedge 1].$$

$\mathcal{B}(\Omega)$  is the Borel  $\sigma$ -algebra of  $\Omega$ .

For each  $t \in [0, \infty)$ , we introduce the following spaces.

- $\Omega_t := \{\omega(\cdot \wedge t) : \omega \in \Omega\}$ ,  $\mathcal{F}_t := \mathcal{B}(\Omega_t)$ ,
- $L^0(\Omega)$  : the space of all  $\mathcal{B}(\Omega)$ -measurable real functions,
- $L^0(\Omega_t)$  : the space of all  $\mathcal{F}_t$ -measurable real functions,
- $B_b(\Omega)$  : all bounded elements in  $L^0(\Omega)$ ,  $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$ ,
- $C_b(\Omega)$  : all continuous elements in  $B_b(\Omega)$ ,  $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$ .

In Peng [23], a  $G$ -Brownian motion is constructed on a sublinear expectation space  $(\Omega, L_G^1, \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t \geq 0})$ , where  $L_G^p(\Omega)$  is a Banach space under the natural norm  $\|X\|_p = \hat{\mathbb{E}}[|X|^p]^{1/p}$ . In this space the corresponding canonical process  $B_t(\omega) = \omega_t$  is a  $G$ -Brownian motion. Denote  $L_b^p(\Omega)$  the completion of  $B_b(\Omega)$ . Denis et al.[2] proved that  $L_b^p(\Omega) \supset L_G^p(\Omega) \supset C_b(\Omega)$ , and there exists a weakly compact family  $\mathcal{P}$  of probability measures defined on  $(\Omega, \mathcal{B}(\Omega))$  such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad X \in L_G^1(\Omega).$$

**Remark 2.1.** Denis et al. [2] gave a concrete set  $\mathcal{P}_M$  that represents  $\hat{\mathbb{E}}$ . Consider a 1-dimensional Brownian motion  $B_t$  on  $(\Omega, \mathcal{F}, P)$ , then

$$\mathcal{P}_M := \{P_\theta : P_\theta = P \circ X^{-1}, X_t = \int_0^t \theta_s dB_s, \theta \in L_{\mathcal{F}}^2([0, T]; [\underline{\sigma}^2, \bar{\sigma}^2])\}$$

is a set that represents  $\hat{\mathbb{E}}$ , where  $L_{\mathcal{F}}^2([0, T]; [\underline{\sigma}^2, \bar{\sigma}^2])$  is the collection of all  $\mathcal{F}$ -adapted measurable processes with  $\underline{\sigma}^2 \leq |\theta(s)|^2 \leq \bar{\sigma}^2$ .

Now we introduce the natural Choquet capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

**Definition 2.2.** A set  $A \subset \mathcal{B}(\Omega)$  is polar if  $c(A) = 0$ . A property holds “quasi-surely” (q.s.) if it holds outside a polar set.

Let  $T \in \mathbb{R}^+$  be fixed.

**Definition 2.3.** For each  $p \geq 1$ , consider the following simple type of processes:

$$M_G^{0,p}([0, T]) = \{\eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t) \\ \forall N > 0, 0 = t_0 < \dots < t_N = T, \xi_j \in \mathbb{L}_G^p(\Omega_{t_j}), j = 0, 1, 2, \dots, N-1\}.$$

Denote by  $M_G^p([0, T])$  the completion of  $M_G^{0,p}([0, T])$  under the norm

$$\|\eta\|_{M_G^p([0, T])} = \left| \int_0^T \hat{\mathbb{E}}[|\eta(t)|^p] dt \right|^{1/p}.$$

Unlike the classical theory, the quadratic variation process of  $G$ -Brownian motion  $B$  is not always a deterministic process and it can be formulated in  $L_G^2(\Omega_t)$  by

$$\langle B \rangle_t := \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} (B_{t_{i+1}^N} - B_{t_i^N})^2 = B_t^2 - 2 \int_0^t B_s dB_s,$$

where  $t_i^N = \frac{iT}{N}$  for each integer  $N \geq 1$ .

Peng [23] also introduced the related stochastic calculus of Itô's type with respect to  $G$ -Brownian motion and the quadratic variation process (see also Li and Peng [12]), i.e.,  $(\int_0^t \eta_s dB_s)_{0 \leq t \leq T}$  and  $(\int_0^t \xi_s d\langle B \rangle_s)_{0 \leq t \leq T}$  are well defined for each  $\eta \in M_G^2([0, T])$  and  $\xi \in M_G^1([0, T])$ .

In view of the dual formulation of  $G$ -expectation as well as the properties of the quadratic variation process  $\langle B \rangle$  in  $G$ -framework, Gao [6] obtained the following BDG type inequalities.

**Lemma 2.4.** *For each  $p \geq 1$  and  $\eta \in M_G^p([0, T])$ ,*

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \eta_s d\langle B \rangle_s \right|^p\right] \leq \bar{\sigma}^{2p} T^{p-1} \int_0^T \hat{\mathbb{E}}[|\eta_s|^p] ds.$$

**Lemma 2.5.** *Let  $p \geq 2$  and  $\eta \in M_G^p([0, T])$ . Then there exists some constant  $C_p$  depending only on  $p$  and  $T$  such that*

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \eta_s dB_s \right|^p\right] \leq C_p \hat{\mathbb{E}}\left[\int_0^T |\eta_s|^2 ds\right]^{\frac{p}{2}}.$$

Recently Lin [13] established stochastic integrals with respect to an increasing process in a Riemann-Stieltjes way, i.e.,  $(\int_0^t X_s dK_s)_{0 \leq t \leq T}$  is well defined for each  $X \in M_c([0, T])$  and  $K \in M_I([0, T])$ , where

- $M_c([0, T])$ : the space of all processes  $X$  whose paths are continuous in  $t$  on  $[0, T]$  outside a polar set  $A$ .
- $M_I([0, T])$ : the space of all q.s. increasing processes  $K \in M_c([0, T])$ .

Moreover an extension of  $G$ -Itô's formula was also obtained. For more details, we refer the reader to Lin [13], [14]. In particular, we recall the following argument.

**Lemma 2.6.** *For some  $p > 2$ , consider a q.s. continuous  $G$ -Itô process  $Y$  defined in the following form*

$$(2.1) \quad Y_t = x + \int_0^t f_s ds + \int_0^t h_s d\langle B \rangle_s + \int_0^t g_s dB_s, \quad 0 \leq t \leq T,$$

where  $f$ ,  $h$  and  $g$  are elements in  $M_G^p([0, T])$ . Then, there exists a unique pair of processes  $(X, K)$  in  $M_G^p([0, T]) \times (M_I([0, T]) \cap M_G^p([0, T]))$  such that

$$(2.2) \quad X_t = Y_t + K_t, \quad q.s.,$$

and (a)  $X$  is positive; (b)  $K_0 = 0$ ; and (c)  $\int_0^T X_t dK_t = 0$ , q.s..

### 3. SCALAR VALUED RGSDES WITH NONLINEAR RESISTANCE

In this section, we give the existence and uniqueness of the solutions to the scalar valued RGSDEs with nonlinear resistance under an integral-Lipschitz condition. Moreover, a comparison theorem is obtained. Without loss of generality, we always assume  $\bar{\sigma}^2 = 1$  in what follows. Before we move to our main results, we want to mention the so-called Bihari's inequality (cf. Bihari [1]).

**Lemma 3.1.** *Let  $\rho : (0, +\infty) \rightarrow (0, +\infty)$  be a continuous and increasing function that vanishes at  $0_+$  and satisfies  $\int_0^1 \frac{dr}{\rho(r)} = +\infty$ . Let  $u$  be a measurable and non-negative function defined on  $(0, +\infty)$  satisfies*

$$u(t) \leq a + \int_0^t \kappa(s) \rho(u(s)) ds, \quad t \in (0, +\infty)$$

where  $a \in \mathbb{R}^+$ , and  $\kappa : [0, T] \rightarrow \mathbb{R}^+$  is Lebesgue integrable. we have

(1) If  $a = 0$ , then  $u(t) = 0$ ,  $t \in (0, +\infty)$ ,  $\lambda$ -a.e.;

(2) If  $a > 0$ , we define

$$v(t) := \int_{t_0}^t \frac{1}{\rho(s)} ds, \quad t \in \mathbb{R}_+,$$

where  $t_0 \in (0, +\infty)$ , then

$$u(t) \leq v^{-1}(v(a) + \int_0^t \kappa(s) ds).$$

We consider the following scalar valued RGSDE with nonlinear resistance:

$$(3.1) \quad X_t = x + \int_0^t f_s(X_s, K_s) ds + \int_0^t h_s(X_s, K_s) d\langle B \rangle_s + \int_0^t g_s(X_s, K_s) dB_s + K_t, \quad 0 \leq t \leq T,$$

where

- (A1) The initial condition  $x \in \mathbb{R}$ ;
- (A2) For some  $p > 2$ , the coefficients  $f$ ,  $h$  and  $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are given functions satisfying for each  $x, y \in \mathbb{R}$ ,  $f_t(x, y)$ ,  $h_t(x, y)$ , and  $g_t(x, y) \in M_G^p([0, T])$  and  $|f_t(x, y)|^p + |h_t(x, y)|^p + |g_t(x, y)|^p \leq |\beta_1(t)|^p + \beta_2^p(|x|^p + |y|^p)$ , where  $\beta_1 \in M_G^p([0, T])$  and  $\beta_2 \in \mathbb{R}_+$ ;
- (A3) The coefficients  $f$ ,  $h$  and  $g$  satisfying an integral-Lipschitz condition, i.e., for each  $t \in [0, T]$  and  $x, x', y, y' \in \mathbb{R}$ ,  $|f_t(x, y) - f_t(x', y')|^p + |h_t(x, y) - h_t(x', y')|^p + |g_t(x, y) - g_t(x', y')|^p \leq \beta(t) \rho(|x - x'|^p + |y - y'|^p)$ , where  $\beta : [0, T] \rightarrow \mathbb{R}^+$  is integrable, and  $\rho : (0, +\infty) \rightarrow (0, +\infty)$  is continuous increasing and concave function that vanishes at  $0_+$  and satisfies

$$\int_0^1 \frac{dr}{\rho(r)} = +\infty;$$

- (A4) The obstacle is a  $G$ -Itô process whose coefficients are all elements in  $M_G^p([0, T])$ , and we shall always assume that  $S_0 \leq x$ , q.s..

The solution of RGSDE with nonlinear resistance (3.1) is a pair of processes  $(X, K)$  which take values both in  $\mathbb{R}$  and satisfy:

- (A5)  $X \in M_G^p([0, T])$  and  $X_t \geq S_t$ ,  $0 \leq t \leq T$ , q.s.;
- (A6)  $K \in M_I([0, T]) \cap M_G^p([0, T])$  and  $K_0 = 0$ , q.s.;
- (A7)  $\int_0^T (X_t - S_t) dK_t = 0$ , q.s..

Now we give our main results.

**Theorem 3.2.** *Let the assumptions (A1)-(A4) hold true, then the RGSDE (3.1) admits a unique solution in  $M_G^p([0, T])$ .*

In order to prove Theorem 3.2, we need some lemmas which give a prior estimate of the solution and estimate of variation in the solutions.

**Lemma 3.3.** *Assume that  $(X, K)$  is a solution to (3.1), then there exists a constant  $C > 0$  such that*

$$\hat{\mathbb{E}}[\sup_{0 \leq s \leq T} |X_t|^p] + \hat{\mathbb{E}}[|K_T|^p] \leq C(|x|^p + \int_0^T \hat{\mathbb{E}}[|\beta_1(t)|^p] dt + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |S_t^+|^p]).$$

**Proof :** Let  $(X, K)$  be a pair of solution to (3.1). Replacing  $Y_t$  by  $x + \int_0^t f_s(X_s, K_s)ds + \int_0^t h_s(X_s, K_s)d\langle B \rangle_s + \int_0^t g_s(X_s, K_s)dB_s - S_t$  and  $X_t$  by  $X_t - S_t$  in (2.2), we have the following representation of  $K$  on  $[0, T]$ :

$$(3.2) \quad K_t = \sup_{0 \leq s \leq t} \left( x + \int_0^s f_u(X_u, K_u)du + \int_0^s h_u(X_u, K_u)d\langle B \rangle_u + \int_0^s g_u(X_u, K_u)dB_u - S_s \right)^-, \quad q.s..$$

As  $X$  is the solution to (3.1), we obtain

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X_s|^p] &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |x + \int_0^s f_u(X_u, K_u)du + \int_0^s h_u(X_u, K_u)d\langle B \rangle_u + \int_0^s g_u(X_u, K_u)dB_u + K_s|^p] \\ &\leq C(|x|^p + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s f_u(X_u, K_u)du|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s h_u(X_u, K_u)d\langle B \rangle_u|^p] \\ &\quad + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s g_u(X_u, K_u)dB_u|^p] + \hat{\mathbb{E}}[|K_t|^p]). \end{aligned}$$

Similarly, from the representation of  $K$  (3.2), we have

$$\begin{aligned} \hat{\mathbb{E}}[K_t^p] &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} ((x + \int_0^s f_u(X_u, K_u)du + \int_0^s h_u(X_u, K_u)d\langle B \rangle_u + \int_0^s g_u(X_u, K_u)dB_u - S_s)^-)^p] \\ &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} ((x + \int_0^s f_u(X_u, K_u)du + \int_0^s h_u(X_u, K_u)d\langle B \rangle_u + \int_0^s g_u(X_u, K_u)dB_u - S_s^+)^-)^p] \\ &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |x + \int_0^s f_u(X_u, K_u)du + \int_0^s h_u(X_u, K_u)d\langle B \rangle_u + \int_0^s g_u(X_u, K_u)dB_u - S_s^+|^p] \\ &\leq C(|x|^p + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s f_u(X_u, K_u)du|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s h_u(X_u, K_u)d\langle B \rangle_u|^p] \\ &\quad + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\int_0^s g_u(X_u, K_u)dB_u|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |S_s^+|^p]). \end{aligned}$$

Combining the above two inequalities and applying BDG type inequalities, we get

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X_s|^p] + \hat{\mathbb{E}}[K_t^p] &\leq C(|x|^p + \int_0^t (\hat{\mathbb{E}}[|f_s(X_s, K_s)|^p] \\ &\quad + \hat{\mathbb{E}}[|h_s(X_s, K_s)|^p] + \hat{\mathbb{E}}[|g_s(X_s, K_s)|^p])ds + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |S_s^+|^p]). \end{aligned}$$

By condition (A2), we deduce

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X_s|^p] + \hat{\mathbb{E}}[K_t^p] &\leq C(|x|^p + \int_0^t (\hat{\mathbb{E}}[|\beta_1(s)|^p + \beta_2^p(|X_s|^p + |K_s|^p)])ds + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |S_s^+|^p]) \\ &\leq C(|x|^p + \int_0^T \hat{\mathbb{E}}[|\beta_1(s)|^p]dt + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |S_t^+|^p] \\ &\quad + \beta_2^p \int_0^t \hat{\mathbb{E}}[\sup_{0 \leq u \leq s} |X_u|^p] + \hat{\mathbb{E}}[|K_s|^p]ds). \end{aligned}$$

Applying Gronwall's lemma to  $\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X_s|^p] + \hat{\mathbb{E}}[|K_t|^p]$ , the result follows.  $\square$

**Lemma 3.4.** Assume that  $(x^i, f^i, h^i, g^i, S^i)$  satisfy (A1)-(A4), and let  $(X^i, K^i)$  be the solution to the RGSDE corresponding to  $(x^i, f^i, h^i, g^i, S^i)$ ,  $i = 1, 2$ . Define

$$\begin{aligned} \Delta x &= x^1 - x^2, \quad \Delta f = f^1 - f^2, \quad \Delta h = h^1 - h^2, \quad \Delta g = g^1 - g^2; \\ \Delta S &= S^1 - S^2, \quad \Delta X = X^1 - X^2, \quad \Delta K = K^1 - K^2. \end{aligned}$$

Then either  $\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\Delta X_t|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq T} |\Delta K_t|^p] = 0$  or there exists a constant  $C > 0$  such that

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\Delta X_t|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq T} |\Delta K_t|^p] &\leq v^{-1} (v(C(|\Delta x|^p + \int_0^T (\hat{\mathbb{E}}[|\Delta f_t(X_t^1, K_t^1)|^p] + \hat{\mathbb{E}}[|\Delta h_t(X_t^1, K_t^1)|^p] \\ &\quad + \hat{\mathbb{E}}[|\Delta g_t(X_t^1, K_t^1)|^p])dt + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\Delta S_t|^p])) \\ &\quad + C \int_0^T \beta(t)dt). \end{aligned}$$

**Proof :** We set

$$(M^X)_t^i = x^i + \int_0^t f_s^i(X_s^i, K_s^i)ds + \int_0^t h_s^i(X_s^i, K_s^i)d\langle B \rangle_s + \int_0^t g_s^i(X_s^i, K_s^i)dB_s, \quad 0 \leq t \leq T, \quad i = 1, 2,$$

and  $\Delta M^X = (M^X)^1 - (M^X)^2$ . By condition (A3), we have

$$\begin{aligned}
\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s|^p] &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta x + \int_0^s (f_u^1(X_u^1, K_u^1) - f_u^2(X_u^2, K_u^2)) du \\
&\quad + \int_0^s (h_u^1(X_u^1, K_u^1) - h_u^2(X_u^2, K_u^2)) d\langle B \rangle_u + \int_0^s (g_u^1(X_u^1, K_u^1) - g_u^2(X_u^2, K_u^2)) dB_u|^p] \\
&\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta x + \int_0^s \Delta f_u(X_u^1, K_u^1) du + \int_0^s (f_u^2(X_u^1, K_u^1) - f_u^2(X_u^2, K_u^2)) du \\
&\quad + \int_0^s \Delta h_u(X_u^1, K_u^1) d\langle B \rangle_u + \int_0^s (h_u^2(X_u^1, K_u^1) - h_u^2(X_u^2, K_u^2)) d\langle B \rangle_u \\
&\quad + \int_0^s \Delta g_u(X_u^1, K_u^1) dB_u + \int_0^s (g_u^2(X_u^1, K_u^1) - g_u^2(X_u^2, K_u^2)) dB_u|^p] \\
&\leq C(|\Delta x|^p + \int_0^t (\hat{\mathbb{E}}[|\Delta f_s(X_s^1, K_s^1)|^p] + \hat{\mathbb{E}}[|\Delta h_s(X_s^1, K_s^1)|^p] \\
&\quad + \hat{\mathbb{E}}[|\Delta g_s(X_s^1, K_s^1)|^p]) ds + \int_0^t \beta(s) \rho(\hat{\mathbb{E}}[|\Delta X_s|^p] + \hat{\mathbb{E}}[|\Delta K_s|^p]) ds),
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta K_s|^p] &= \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\sup_{0 \leq u \leq s} ((M^X)_u^1 - S_u^1)^- - \sup_{0 \leq u \leq s} ((M^X)_u^2 - S_u^2)^-|^p] \\
&\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\sup_{0 \leq u \leq s} |((M^X)_u^1 - S_u^1)^- - ((M^X)_u^2 - S_u^2)^-|^p] \\
&= \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |((M^X)_s^1 - S_s^1)^- - ((M^X)_s^2 - S_s^2)^-|^p] \\
&\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |((M^X)_s^1 - S_s^1) - ((M^X)_s^2 - S_s^2)|^p] \\
&\leq C(\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta(M^X)_s|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta S_s|^p]).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta X_s|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta K_s|^p] &\leq \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s + \Delta K_s|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta K_s|^p] \\
&\leq C(\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s|^p] + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta K_s|^p]) \\
&\leq C(|\Delta x|^p + \int_0^t (\hat{\mathbb{E}}[|\Delta f_s(X_s^1, K_s^1)|^p] + \hat{\mathbb{E}}[|\Delta h_s(X_s^1, K_s^1)|^p] \\
&\quad + \hat{\mathbb{E}}[|\Delta g_s(X_s^1, K_s^1)|^p]) ds + \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |\Delta S_s|^p] \\
&\quad + \int_0^t \beta(s) \rho(\hat{\mathbb{E}}[|\Delta X_s|^p] + \hat{\mathbb{E}}[|\Delta K_s|^p]) ds) \\
&\leq C(|\Delta x|^p + \int_0^T (\hat{\mathbb{E}}[|\Delta f_t(X_t^1, K_t^1)|^p] + \hat{\mathbb{E}}[|\Delta h_t(X_t^1, K_t^1)|^p] \\
&\quad + \hat{\mathbb{E}}[|\Delta g_t(X_t^1, K_t^1)|^p]) dt + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\Delta S_t|^p] \\
&\quad + \int_0^t \beta(s) \rho(\hat{\mathbb{E}}[\sup_{0 \leq u \leq s} |\Delta X_u|^p] + \hat{\mathbb{E}}[\sup_{0 \leq u \leq s} |\Delta K_u|^p]) ds).
\end{aligned}$$

Lemma 3.1 gives the desired result.  $\square$

Now we will give the proof of Theorem 3.2.

**Proof of Theorem 3.2:** By taking  $x^1 = x^2$ ,  $f^1 = f^2$ ,  $h^1 = h^2$ ,  $g^1 = g^2$  and  $S^1 = S^2$  in Lemma 3.4, we obtain immediately the uniqueness result. Now we will establish the existence result for RGSDE with nonlinear resistance (3.1) by using a Picard iteration. Letting  $X^0 = x$  and  $K^0 = 0$ , for each  $n \in \mathbb{N}_+$ ,  $X^{n+1}$  and  $K^{n+1}$  are given by recurrence:

$$X_t^{n+1} = x + \int_0^t f_s(X_s^n, K_s^n) ds + \int_0^t h_s(X_s^n, K_s^n) d\langle B \rangle_s + \int_0^t g_s(X_s^n, K_s^n) dB_s + K_t^{n+1}, \quad 0 \leq t \leq T,$$

satisfying

- (a)  $X_t^{n+1} \in M_G^p([0, T])$ ,  $X_t^{n+1} \geq S_t$ ,  $q.s.$ ;
- (b)  $K_t^{n+1} \in M_I([0, T]) \cap M_G^p([0, T])$ ,  $K_t^{n+1} = 0$ ,  $q.s.$ ;
- (c)  $\int_0^T (X_t^{n+1} - S_t) dK_t^{n+1} = 0$ ,  $q.s.$

Actually  $X_t^{n+1} = \tilde{X}_t^{n+1}$  and  $K_t^{n+1} = \tilde{K}_t^{n+1}$ , where  $(\tilde{X}^{n+1}, \tilde{K}^{n+1})$  is given by Lemma 2.6 with  $Y_t = x + \int_0^t f_s(X_s^n, K_s^n) ds + \int_0^t h_s(X_s^n, K_s^n) d\langle B \rangle_s + \int_0^t g_s(X_s^n, K_s^n) dB_s - S_t$ . Thus  $(X^{n+1}, K^{n+1})$  is well defined in  $M_G^p([0, T]) \times (M_I([0, T]) \cap M_G^p([0, T]))$ .

Similarly to Lemma 3.3, we get a priori estimate uniform in  $n$  for  $\{\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |X_t^n|^p] + \hat{\mathbb{E}}[|K_T^n|^p]\}_{n \in \mathbb{N}}$ . Indeed, we have

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |X_t^{n+1}|^p\right] + \hat{\mathbb{E}}[|K_t^{n+1}|^p] &\leq C \left( |x|^p + \int_0^t \hat{\mathbb{E}}[|\beta_1(s)|^p] ds + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |S_s^+|^p\right] \right. \\ &\quad \left. + \int_0^t \beta_2^p (\hat{\mathbb{E}}\left[\sup_{0 \leq u \leq s} |X_u^n|^p\right] + \hat{\mathbb{E}}[|K_s^n|^p]) ds \right). \end{aligned}$$

Set

$$a(t) := Ce^{C\beta_2^p t} \left( |x|^p + \int_0^T \hat{\mathbb{E}}[|\beta_1(s)|^p] ds + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq T} |S_s^+|^p\right] \right),$$

then  $a(\cdot)$  is the solution to the following ordinary differential equation:

$$a(t) = C \left( |x|^p + \int_0^T \hat{\mathbb{E}}[|\beta_1(s)|^p] ds + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq T} |S_s^+|^p\right] + \beta_2^p \int_0^t a(s) ds \right).$$

It is easy to check that for all  $n \in \mathbb{N}$ ,  $\hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X_t^n|^p] + \hat{\mathbb{E}}[|K_t^n|^p] \leq a(t)$ .

Secondly, for  $n$  and  $m \in \mathbb{N}$ , we define

$$u_t^{n+1,m} := \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |X_s^{n+m+1} - X_s^{n+1}|^p\right] + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |K_s^{n+m+1} - K_s^{n+1}|^p\right], \quad 0 \leq t \leq T.$$

Then it follows from the same techniques of the proof of Theorem 3.7 in Lin and Bai [16] that  $\{X^n\}_{n \in \mathbb{N}}$  and  $\{K^n\}_{n \in \mathbb{N}}$  are two Cauchy sequences in  $M_G^p([0, T])$ . We denote the limit by  $X$  and  $K$ . Following the procedures of the proof of Lemma 3.4 and noting that  $\rho$  is continuous and  $\rho(0_+) = 0$ ,  $K$  has the representation (3.2).

Obviously, the pair of processes  $(X, K)$  satisfies (A5) - (A7). Thus the pair of processes  $(X, K)$ , well defined in  $M_G^p([0, T]) \times (M_I([0, T]) \cap M_G^p([0, T]))$ , is a solution to (3.1).  $\square$

In order to give the comparison principle, we consider the following scalar valued RGSDE:

$$X_t = x + \int_0^t f_s(X_s, K_s) ds + \int_0^t h_s(X_s, K_s) d\langle B \rangle_s + \int_0^t g_s(X_s) dB_s + K_t, \quad 0 \leq t \leq T,$$

and we assume that:

- (A2') For some  $p > 2$ , the coefficients  $f, h: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are given functions satisfying for each  $x, y \in \mathbb{R}$ ,  $f_t(x, y), h_t(x, y)$ , and  $g_t(x) \in M_G^p([0, T])$  and  $|f_t(x, y)|^p + |h_t(x, y)|^p + |g_t(x)|^p \leq |\beta_1(t)|^p + \beta_2^p(|x|^p + |y|^p)$ , where  $\beta_1 \in M_G^p([0, T])$  and  $\beta_2 \in \mathbb{R}_+$ ;
- (A3') The coefficients  $f, h$  and  $g$  satisfying an integral-Lipschitz condition, i.e., for each  $t \in [0, T]$  and  $x, x', y, y' \in \mathbb{R}$ ,  $|f_t(x, y) - f_t(x', y')|^p + |h_t(x, y) - h_t(x', y')|^p + |g_t(x) - g_t(x')|^p \leq \rho(|x - x'|^p + |y - y'|^p)$ , where  $\rho: (0, +\infty) \rightarrow (0, +\infty)$  is continuous increasing and concave function that vanishes at  $0_+$  and satisfies

$$\int_0^1 \frac{dr}{r + \rho(r)} = +\infty.$$

**Remark 3.5.** Before we move to the comparison principle, we should mention that restricted to the fact that we need to apply G-Itô's formula and we need to consider  $X$  separately, we should impose stronger conditions on the coefficients but which are still weaker than those assumed in Lin [13]. It is easy to check that (A3') implies (A3). Following the comparison result in Lin [13], at first, we assume that coefficients  $f, h$  and  $g$  and the obstacle process  $S$  are bounded, and then we remove it in the second step.

We then have the following results.

**Theorem 3.6.** Suppose that for  $i = 1, 2$ ,  $f^i, h^i, g^i$  satisfy the conditions (A1), (A2'), (A3') and (A4), and we assume the following:

- (1)  $x^1 \leq x^2$ ;  
 (2)  $f^i, h^i$  and  $g^1 = g^2 = g$  are bounded, and  $S^i$  are uniformly upper bounded,  $i = 1, 2$ ;  
 (3)  $f_t^1(x, 0) \leq f_t^2(x, 0)$  and  $h_t^1(x, 0) \leq h_t^2(x, 0)$ , for  $x \in \mathbb{R}$ ,  $f^1, h^1$  are decreasing in  $y$ , and  $f^2, h^2$  are increasing in  $y$ , and  $S_t^1 \leq S_t^2$ ,  $0 \leq t \leq T$ , *q.s.*  
 If  $(X^i, K^i)$  is the solution to the RGSDEs with data  $(f^i, h^i, g, S^i)$ ,  $i = 1, 2$ , then,

$$X_t^1 \leq X_t^2, \quad 0 \leq t \leq T, \quad \text{q.s.}$$

**Proof :** Since  $f^i, h^i$  and  $g$  are bounded, and  $S^i$  are uniformly upper bounded,  $i = 1, 2$ , using the BDG type inequalities to (3.2), we deduce that  $K_T^i$  has the moment for arbitrage large order and for  $0 \leq t \leq T$ ,  $\lim_{s \rightarrow t} \mathbb{E}[K_t^i - K_s^i]^2] = 0$ ,  $i = 1, 2$ .

Compared to Lin [13], we consider  $(x^+)^p$  and apply the extended  $G$ -Itô's formula to  $((X_t^1 - X_t^2)^+)^p$ ,

$$\begin{aligned}
 ((X_t^1 - X_t^2)^+)^p &= 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (f_s^1(X_s^1, K_s^1) - f_s^2(X_s^2, K_s^2)) ds \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (h_s^1(X_s^1, K_s^1) - h_s^2(X_s^2, K_s^2)) d\langle B \rangle_s \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (g_s(X_s^1) - g_s(X_s^2)) dB_s \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} d(K_s^1 - K_s^2) \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-2} |g_s(X_s^1) - g_s(X_s^2)|^2 d\langle B \rangle_s
 \end{aligned}
 \tag{3.3}$$

Since on  $\{X_t^1 > X_t^2\}$ ,  $X_t^1 > X_t^2 \geq S_t^2 \geq S_t^1$ , we have

$$\begin{aligned}
 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} d(K_s^1 - K_s^2) &= \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} dK_s^1 - \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} dK_s^2 \\
 &\leq \int_0^t |(X_s^1 - S_s^1)^+|^{p-1} dK_s^1 - \int_0^t |(X_s^1 - X_s^2)^+|^2 dK_s^{p-1} \\
 &\leq - \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} dK_s^2 \leq 0, \quad \text{q.s.}
 \end{aligned}
 \tag{3.4}$$

Noting the monotonicity of  $f^1, h^1, f^2$ , and  $h^2$  and (3.4), we have

$$\begin{aligned}
 ((X_t^1 - X_t^2)^+)^p &\leq 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (f_s^1(X_s^1, 0) - f_s^2(X_s^2, 0)) ds \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (h_s^1(X_s^1, 0) - h_s^2(X_s^2, 0)) d\langle B \rangle_s \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-1} (g_s(X_s^1) - g_s(X_s^2)) dB_s \\
 &\quad + 3 \int_0^t |(X_s^1 - X_s^2)^+|^{p-2} |g_s(X_s^1) - g_s(X_s^2)|^2 d\langle B \rangle_s
 \end{aligned}
 \tag{3.5}$$

Then, by condition (A3'), Young's inequality, taking  $G$ -expectation on both sides of (3.5) and Jensen's inequality, we conclude

$$\begin{aligned}
 \hat{\mathbb{E}}[((X_t^1 - X_t^2)^+)^p] &\leq C \hat{\mathbb{E}} \left[ \int_0^t ((X_s^1 - X_s^2)^+)^p + \rho(((X_s^1 - X_s^2)^+)^p) ds \right] \\
 &\leq C \int_0^t \hat{\mathbb{E}}[((X_s^1 - X_s^2)^+)^p] + \rho(\hat{\mathbb{E}}[((X_s^1 - X_s^2)^+)^p]) ds.
 \end{aligned}$$

Using Biharis inequality, it follows that  $\hat{\mathbb{E}}[((X_t^1 - X_t^2)^+)^p] = 0$ , which implies the result.  $\square$



**Theorem 3.7.** Suppose that for  $i = 1, 2$   $f^i$ ,  $h^i$ ,  $g^i$  satisfy the conditions (A1), (A2'), (A3') and (A4), and we assume in addition the following:

- (1)  $x^1 \leq x^2$  and  $g^1 = g^2 = g$ ;
- (2)  $f_t^1(x, 0) \leq f_t^2(x, 0)$  and  $h_t^1(x, 0) \leq h_t^2(x, 0)$ , for  $x \in \mathbb{R}$ ,  $f^1, h^1$  are decreasing in  $y$ , and  $f^2, h^2$  are increasing in  $y$ , and  $S_t^1 \leq S_t^2$ ,  $0 \leq t \leq T$ ,  $q.s.$ .

If  $(X^1, K^1)$  and  $(X^2, K^2)$  are the solutions to the RGSDEs above respectively, then,

$$X_t^1 \leq X_t^2, \quad 0 \leq t \leq T, \quad q.s..$$

**Proof :** Firstly, we define the truncation functions for the coefficients and the obstacle process: for  $N > 0$ ,  $\xi_t^N(x) = (-N \vee \xi_t(x)) \wedge N$ ,  $x \in \mathbb{R}$ , where  $\xi$  denote  $f^i$ ,  $h^i$  and  $g$ ,  $i = 1, 2$ , and  $S_t^N = S_t \wedge N$ ,  $0 \leq t \leq T$ . It is easy to verify that the truncated coefficients and obstacle processes satisfy (A2') and (A3'). Moreover, the truncation functions keep the order of the coefficients and obstacle processes, that is,

$$(f^1)_t^N(x, 0) \leq (f^2)_t^N(x, 0), \quad (h^1)_t^N(x, 0) \leq (h^2)_t^N(x, 0), \quad \text{and} \quad (S^1)_t^N \leq (S^2)_t^N, \quad 0 \leq t \leq T, \quad q.s..$$

$$(f^1)_t^N, (h^1)_t^N \text{ are decreasing in } y, \quad \text{and} \quad (f^2)_t^N, (h^2)_t^N \text{ are increasing in } y.$$

Consider the following RGSDEs on  $[0, T]$ , for  $i = 1, 2$ ,

$$(X^i)_t^N = x + \int_0^t (f^i)_s^N((X^i)_s^N, (K^i)_s^N) ds + \int_0^t (h^i)_s^N((X^i)_s^N, (K^i)_s^N) d\langle B \rangle_s + \int_0^t g_s^N((X^i)_s^N) dB_s + (K^i)_t^N;$$

satisfies

- (a)  $(X^i)_t^N \in M_G^p([0, T])$ ,  $(X^i)_t^N \geq (S^i)_t^N$ ,  $q.s.$ ;
- (b)  $(K^i)_t^N \in M_I([0, T]) \cap M_G^p([0, T])$ ,  $(K^i)_0^N = 0$ ,  $q.s.$ ;
- (c)  $\int_0^T ((X^i)_t^N - (S^i)_t^N) d(K^i)_t^N = 0$ ,  $q.s..$

By Theorem 3.6, we have

$$(3.6) \quad (X^1)_t^N \leq (X^2)_t^N, \quad q.s..$$

In view of the proof of Lemma 3.4, we have

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |(X^i)_t^N - X_t^i|^p] + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |(K^i)_t^N - K_t^i|^p] &\leq C(\int_0^T (\hat{\mathbb{E}}[|(f^i)_t^N(t, X_t^i, K_t^i) - f^i(t, X_t^i, K_t^i)|^p] \\ &\quad + \hat{\mathbb{E}}[|(h^i)_t^N(t, X_t^i, K_t^i) - h^i(t, X_t^i, K_t^i)|^p] \\ &\quad + \hat{\mathbb{E}}[|(g^i)_t^N(t, X_t^i) - g(t, X_t^i)|^p] dt \\ &\quad + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |(S^i)_t^N - S_t^i|^p]). \end{aligned}$$

For any  $t \in [0, T]$ , by condition (A2') we calculate

$$\begin{aligned} \hat{\mathbb{E}}[|(f^i)_t^N(X_t^i, K_t^i) - f_t^i(X_t^i, K_t^i)|^p] &\leq \hat{\mathbb{E}}[|f_t^i(X_t^i, K_t^i)|^p I_{\{|f_t^i(X_t^i, K_t^i)| > N\}}] \\ &\leq \hat{\mathbb{E}}[(|\beta_1(t)|^p + \beta_2^p(|X_t^i|^p + |K_t^i|^p)) I_{\{(|\beta_1(t)|^p + \beta_2^p(|X_t^i|^p + |K_t^i|^p)) > N^p\}}] \\ &\leq C(\hat{\mathbb{E}}[|\beta_1(t)|^p I_{\{|\beta_1(t)|^p > \frac{N^p}{3}\}}] + \hat{\mathbb{E}}[|X_t^i|^p I_{\{|X_t^i|^p > \frac{N^p}{3\beta_2^p}\}}] + \hat{\mathbb{E}}[|K_t^i|^p I_{\{|K_t^i|^p > \frac{N^p}{3\beta_2^p}\}}]). \end{aligned}$$

Since  $\beta_1(\cdot)$ ,  $X^i$  and  $K^i \in M_G^p([0, T])$ , we obtain we that  $\beta_1(t)$ ,  $X_t^i$  and  $K_t^i \in L_G^p(\Omega_t)$  for almost every  $t \in [0, T]$ . Therefore, letting  $N \rightarrow +\infty$ , we have

$$\hat{\mathbb{E}}[|(f^i)_t^N(X_t^i, K_t^i) - f_t^i(X_t^i, K_t^i)|^p] \rightarrow 0.$$

Similarly, we can obtain that

$$\hat{\mathbb{E}}[|(h^i)_t^N(X_t^i, K_t^i) - h_t^i(X_t^i, K_t^i)|^p] \rightarrow 0;$$

and

$$\hat{\mathbb{E}}[|(g^i)_t^N(X_t^i) - g_t^i(X_t^i)|^p] \rightarrow 0.$$

Thus, it follows from the procedures of the proof of Theorem 5.9 in Lin [13] that

$$(3.7) \quad \hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |(X^i)_t^N - X_t^i|^p\right] \rightarrow 0, \text{ as } N \rightarrow +\infty.$$

Then, (3.6) and (3.7) yield the desired result.  $\square$

As a consequent result of Theorem 3.7, we have the following corollary.

**Corollary 3.8.** *Given two RGSDEs satisfying the conditions (A1), (A2'), (A3') and (A4), and suppose that  $x^1 \leq x^2$  and  $g^1 = g^2 = g$ , if one of the following holds:*

- (1)  *$f^1$  and  $h^1$  are independent of  $y$ , and  $f_t^1(x) \leq f_t^2(x, 0)$  and  $h_t^1(x) \leq h_t^2(x, 0)$ , for  $x \in \mathbb{R}$ ,  $f^2, h^2$  are increasing in  $y$ , and  $S_t^1 \leq S_t^2$ ,  $0 \leq t \leq T$ , q.s.;*
- (2)  *$f^2$  and  $h^2$  are independent of  $y$ , and  $f_t^1(x, 0) \leq f_t^2(x)$  and  $h_t^1(x, 0) \leq h_t^2(x)$ , for  $x \in \mathbb{R}$ ,  $f^1, h^1$  are decreasing in  $y$ , and  $S_t^1 \leq S_t^2$ ,  $0 \leq t \leq T$ , q.s..*

*Let  $(X^1, K^1)$  and  $(X^2, K^2)$  are two pairs of solutions to the RGSDEs above respectively, then,*

$$X_t^1 \leq X_t^2, \quad 0 \leq t \leq T, \quad \text{q.s..}$$

## Acknowledgement

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